CLOSED STRING WITH MASSES
IN MODELS OF BARYONS AND GLUEBALLS

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Abstract

The closed string carrying n point-like masses is considered as the model of a baryon ($n = 3$), a glueball ($n = 2$ or 3) or another exotic hadron. For this system the rotational states are obtained and classified. They correspond to exact solutions of dynamical equations, describing an uniform rotation of the string with massive points. These rotational states result in a set of quasilinear Regge trajectories with different behavior.

The stability problem for the so called central rotational states (with a mass at the rotational center) is solved with using the analysis of small disturbances. These states turned out to be unstable, if the central mass is less than some critical value.

1. Introduction

The Nambu-Goto string (or relativistic string) simulates strong interaction between quarks at large distances in various string models of mesons and baryons [1] – [12] (Fig. 1a – e). This string has linearly growing energy (energy density is equal to the string tension γ) and describes nonperturbative contribution of the gluon field and the QCD confinement mechanism.

Such a string with massive ends in Fig. 1a may be regarded as the meson string model [2, 3]. String models of the baryon were suggested in the following four topologically different variants [4]: (b) the quark-diquark model q-qq [6] (on the classic level it coincides with the meson model (a), (c) the linear configuration q-q-q [7], (d) the "three-string" model or Y configuration [4, 8], and (e) the "triangle" model or Δ configuration [9, 10].

All cited string hadron models generate linear or quasilinear Regge trajectories in the limit of large energies for excited states of mesons and baryons [5, 6, 11, 12]

$$J \simeq \alpha_0 + \alpha' E^2, \tag{1}$$

if we use rotational states of these systems (planar uniform rotations). Here J and E are the angular momentum and energy of a hadron state (or rotational state of a string model), the slope $\alpha' \simeq 0.9$ GeV^{-2}. For the model of meson in Fig. 1a and for the quark-diquark baryon model in Fig. 1b this slope and the string tension γ are connected by the Nambu relation [1]

$$\alpha' = \frac{1}{2\pi\gamma}. \tag{2}$$

The same relation (2) takes place for the linear baryon configuration (Fig. 1c) in the case of its central rotational states. In these states the middle mass is at the rotational center. In
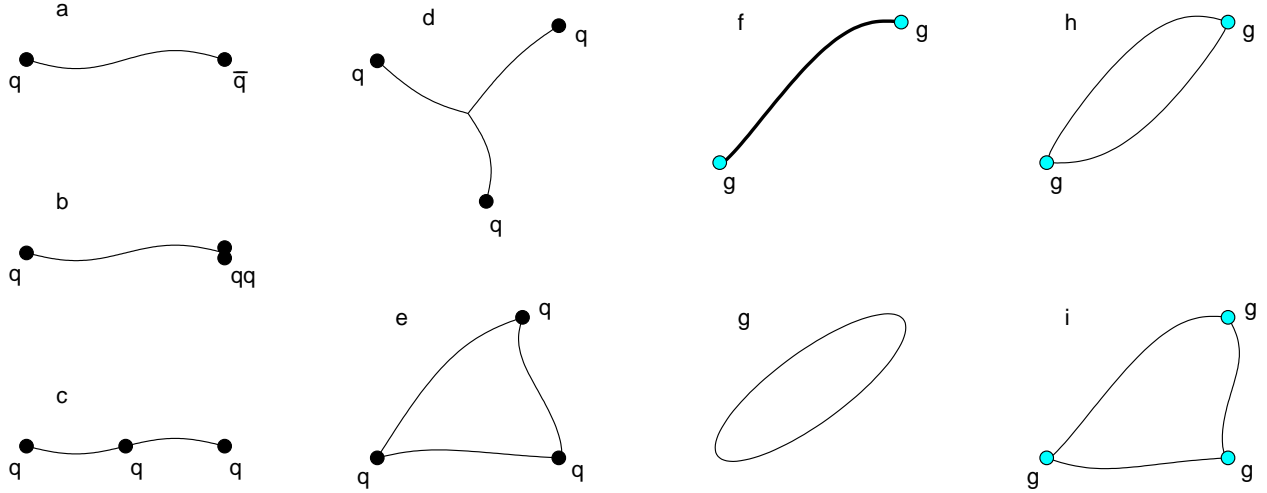
Figure 1: String models of mesons, baryons and glueballs

papers [7, 13] we have shown that the mentioned states are unstable with respect to small disturbances.

The string baryon model Y (Fig. 1d) for its rotational states demonstrates the Regge asymptotics (1) with the slope [8] $\alpha' = 1/(3\pi\gamma)$. To describe the experimental Regge trajectories with the slope $\alpha' \simeq 0.9$ GeV^{-2} we are to assume that the effective string tension γ_Y in this model differs from the value γ in models in Figs. 1a – c (the fundamental string tension) [5, 11] and equals $\gamma_Y = \frac{2}{3}\gamma$. Moreover, the rotations of the Y string configuration are also unstable with respect to small disturbances on the classic level [13, 14].

In this paper we'll concentrate on applications of the closed string carrying n point-like masses in hadron spectroscopy. The string baryon model "triangle" or Δ is the particular case of this system if $n = 3$. This model generates a set of rotational states with different topology [9, 10]. The so called triangle states was applied for describing excited baryon states on the Regge trajectories [5, 11], but in this case (like for the model Y) we are to take another effective string tension $\gamma_\Delta = \frac{3}{8}\gamma$.

Different string models were used for describing glueballs (bound states of gluons) [15] – [25] and other exotic hadrons [26]. There are a lot of experimentally observed hadron states which may be interpreted as glueballs [27, 28] predicted in QCD. But these states can mix with meson states, so their glueball interpretation is ambiguous one.

String models of glueballs or some exotic hadron states (glueball candidates) were suggested in the following variants [15] – [25] shown in Fig. 1: (f) the open string with enhanced tension (the adjoint string) and two constintuent gluons at the endpoints; (g) the closed string simulating gluonic field; (h) the closed string carrying two point-like masses (constituent gluons). Evidently, the last model may be easily generalized for three-gluon glueballs [25] in Fig. 1i. The cited authors suggested different approaches. Some of them [21] – [25] used potential models with string term in the potential.

Glueballs, their masses and momenta, corresponding Regge trajectories are simulated in lattice calculations [29] – [32]. In this approach the QCD glueball may be identified with the pomeron that is the Regge pole determining an asymptotic behavior of high-energy diffractive

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amplitudes [32] – [35]. The pomeron Regge trajectory [31, 32, 34]

$$J \simeq 1.08 + 0.25E^2 \tag{3}$$

differs from hadronic ones (1): its slope $\alpha' \simeq 0.25$ is essentially lower, and its intercept $\alpha_0 \simeq 1.08$ is positive and rather large.

The mentioned string models of glueballs in in Fig. 1$f-i$ were suggested, in particular, to describe Regge trajectories of the type (3).

The model in Fig. 1f considered in Refs. [17, 18, 32, 33], is the open string with massive ends (describing gluons). This string (the adjoint string) has the following tension [32, 36]

$$\gamma_{adj} = \frac{2N_c^2}{N_c^2 - 1}\gamma = \frac{9}{4}\gamma.$$

It exceeds tension γ in hadron models in Fig. 1$a-c$ (fundamental string). In accordance with the Nambu relation (2) the corresponding slope of Regge trajectories is $\frac{4}{9}$ of this value for mesons: $\alpha' \simeq 0.4$. It is larger than the slope $\alpha' \simeq 0.25$ of the trajectory (3).

In papers [15, 19, 20, 31, 32] the closed string without masses (Fig. 1g) is used for describing glueballs. The authors chose different classes of classic motions (states) for modelling trajectories (3) for glueballs. In Ref. [15] this class includes rotations and oscillations of the elliptic closed Nambu-Goto string. This results to a set of slopes of Regge trajectories from 0 to $(4\pi\gamma)^{-1}$. In Refs. [31, 32] the circular shape of the string is fixed and the spectrum of its excitations is considered. In Refs. [19, 20, 37] the closed string is embedded into spaces with nontrivial geometry.

The closed string carrying two massive points (Fig. 1h) (they describe constituent gluons) in Ref. [21] is taken as a basis of the potential model of a glueball. The similar potential model of the 3-gluon glueball is constructed in Ref. [25]. It corresponds to the system in Fig. 1i. The latter model is similar to the string baryon model "triangle" [9, 10] in Fig. 1e. [9, 10]. But the configurations in Figs. 1h and i are not investigated yet as string models of a glueball.

In this paper we consider classical dynamics of the closed string carrying n massive points (generalization of the models in Figs. 1e, h and i) in Minkowski space $R^{1,3}$. In Sect. 3 rotational states (planar uniform rotations) of this system are described and classified. They have much more complicated structure than a well known set of rotations of the folded rectilinear string. All rotational states are divided into 3 classes: linear, hypocycloidal and central states (in the last case a mass is at the rotational center). In Sect. 4 the stability problem for central rotational states is solved with using the analysis of small disturbances. Rotational states of string systems are widely used for generating Regge trajectories. Their structure and behavior for the considered system are described in Sect. 5.

2. Dynamics

The dynamics of the closed string carrying n point-like masses m_1, m_2, ... m_n is determined by the action

$$S = -\gamma \int\limits_{\Omega} \sqrt{-g}\, d\tau d\sigma - \sum_{i=1}^{n} m_i \int \sqrt{\dot{x}_i^2(\tau)}\, d\tau, \tag{4}$$

generalizing the case of the string baryon model "triangle" [10]. Here γ is the string tension, g is the determinant of the induced metric $g_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ on the string world surface $X^\mu(\tau, \sigma)$, the speed of light $c = 1$. The world surface mapping in $R^{1,3}$ from $\Omega = \{\tau, \sigma : \tau_1 < \tau <$

τ_2, $\sigma_0(\tau) < \sigma < \sigma_n(\tau)\}$ is divided by the world lines of massive points $x_i^\mu(\tau) = X^\mu(\tau, \sigma_i(\tau))$, $i = 0, \ldots, n$ into n world sheets. Two of these functions $x_0(\tau)$ and $x_n(\tau)$ describe the same trajectory of the n-th massive point, and their equality forms the closure condition

$$X^\mu(\tau, \sigma_0(\tau)) = X^\mu(\tau^*, \sigma_n(\tau^*)) \tag{5}$$

on the tube-like world surface [10, 37]. These equations may contain two different parameters τ and τ^*, connected via the relation $\tau^* = \tau^*(\tau)$. This relation should be included in the closure condition (5) of the world surface.

Equations of motion of this system result from the action (4) and its variation. They may be reduced to the simplest form under the orthonormality conditions on the world surface

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0, \tag{6}$$

and the conditions

$$\sigma_0(\tau) = 0, \qquad \sigma_n(\tau) = 2\pi. \tag{7}$$

Conditions (6), (7) always may be fixed without loss of generality, if we choose the relevant coordinates τ, σ [10]. It is connected with the invariance of the action (4) with respect to nondegenerate reparametrizations on the world surface $\tau = \tau(\tilde\tau, \tilde\sigma)$, $\sigma = \sigma(\tilde\tau, \tilde\sigma)$. The scalar square in Eq. (6) results from scalar product $(\xi, \zeta) = \eta_{\mu\nu} \xi^\mu \zeta^\nu$.

The orthonormality conditions (6) are equivalent to the conformal flatness of the induced metric g_{ab}. Under conditions (6), (7) the string motion equations take the form [5, 10, 37]

$$\frac{\partial^2 X^\mu}{\partial\tau^2} - \frac{\partial^2 X^\mu}{\partial\sigma^2} = 0, \tag{8}$$

$$m_i \frac{d}{d\tau} \frac{\dot x_i^\mu(\tau)}{\sqrt{\dot x_i^2(\tau)}} + \gamma\Big[X'^\mu + \dot\sigma_i(\tau)\dot X^\mu\Big]\Big|_{\sigma=\sigma_i-0} - \gamma\Big[X'^\mu + \dot\sigma_i(\tau)\dot X^\mu\Big]\Big|_{\sigma=\sigma_i+0} = 0, \tag{9}$$

$$m_n \frac{d}{d\tau} \frac{\dot x_0^\mu(\tau)}{\sqrt{\dot x_0^2(\tau)}} + \gamma[X'^\mu(\tau^*, \sigma_n) - X'^\mu(\tau, 0)] = 0. \tag{10}$$

Here $i = 1, \ldots, n - 1$, $\dot X^\mu \equiv \partial_\tau X^\mu$, $X'^\mu \equiv \partial_\sigma X^\mu$.

Eqs. (9), (10) are equations of motion for the massive points resulting from the action (4). They may be interpreted as boundary conditions for Eq. (8).

We denote the unit vectors

$$e_0^\mu, \ e_1^\mu, \ e_2^\mu, \ e_3^\mu,$$

associated with coordinates x^μ. These vectors form the orthonormal basis in $R^{1,3}$.

The system of equations $(5) - (10)$ describe dynamics of the closed string carrying n point-like masses without loss of generality. One also should add that the tube-like world surface of the closed string is continuous one, but its derivatives may have discontinuities at the world lines of the massive points (except for derivatives along these lines) [10]. These discontinuities are taken into account in Eqs. (9), (10).

3. Rotational states

We search rotational solutions of system $(5) - (10)$ using the approach supposed in Ref. [10] for the string model "triangle" and in Ref. [37] for the closed string carrying one massive point.

In the frameworks of the orthonormality gauge (6) we suppose that the system uniformly rotates, masses move at constant speeds v_i along circles and conditions

$$\sigma_i(\tau) = \sigma_i = \text{const}, \qquad i = 1, \ldots, n, \tag{11}$$

$$\tau^* = \tau + 2\pi\theta, \qquad \theta = \text{const}, \tag{12}$$

$$\frac{\gamma}{m_i} \sqrt{\dot{X}^2(\tau, \sigma_i)} = Q_i = \text{const}, \qquad i = 1, \ldots, n \tag{13}$$

are fulfilled.

When we search solution of the linearized system $(5)-(10)$ under restrictions $(11)-(13)$ as a linear combination of terms $X^\mu(\tau, \sigma) = T^\mu(\tau)\, u(\sigma)$ (Fourier method) we obtain from Eq. (8) two equations for functions $T^\mu(\tau)$ and $u(\sigma)$:

$$T_\mu''(\tau) + \omega^2 T_\mu = 0, \qquad u''(\sigma) + \omega^2 u = 0.$$

Their solutions describing uniform rotations of the string system (rotational states) contain one nonzero frequency ω and have the following form [10, 37]:

$$X^\mu(\tau, \sigma) = x_0^\mu + e_0^\mu(a_0\tau + b_0\sigma) + u(\sigma) \cdot e^\mu(\omega\tau) + \tilde{u}(\sigma) \cdot \acute{e}^\mu(\omega\tau). \tag{14}$$

Here
$$e^\mu(\omega\tau) = e_1^\mu \cos\omega\tau + e_2^\mu \sin\omega\tau, \qquad \acute{e}^\mu(\omega\tau) = -e_1^\mu \sin\omega\tau + e_2^\mu \cos\omega\tau$$

are unit orthogonal vectors rotating in the plane e_1, e_2; the function

$$u(\sigma) = \begin{cases} A_1 \cos\omega\sigma + B_1 \sin\omega\sigma, & \sigma \in [0, \sigma_1], \\ A_2 \cos\omega\sigma + B_2 \sin\omega\sigma, & \sigma \in [\sigma_1, \sigma_2], \\ \ldots \\ A_n \cos\omega\sigma + B_n \sin\omega\sigma, & \sigma \in [\sigma_{n-1}, 2\pi] \end{cases}$$

and its analog
$$\tilde{u}(\sigma) = \tilde{A}_i \cos\omega\sigma + \tilde{B}_i \sin\omega\sigma, \qquad \sigma \in [\sigma_{i-1}, \sigma_i]$$

are continuous, but their derivatives have discontinuities at $\sigma = \sigma_i$ (positions of masses m_i).

Continuity of functions $u(\sigma)$ and $\tilde{u}(\sigma)$ at $\sigma = \sigma_i$ results in equalities

$$(\mathcal{A}_{i+1} - \mathcal{A}_i) \cos\omega\sigma_i = (\mathcal{B}_i - \mathcal{B}_{i+1}) \sin\omega\sigma_i. \tag{15}$$

Here we use the notations for columns

$$\mathcal{A}_i = \begin{pmatrix} A_i \\ \tilde{A}_i \end{pmatrix}, \qquad \mathcal{B}_i = \begin{pmatrix} B_i \\ \tilde{B}_i \end{pmatrix}. \tag{16}$$

Expression (14) is the solution of Eq. (8) and it must satisfy the conditions (5), (6), (9), (10) under restrictions $(11)-(13)$. Boundary conditions (9) with adding Eq. (13) take the form
$$\ddot{X}(\tau, \sigma_i) + Q_i\left[X'^\mu(\tau, \sigma_i - 0) - X'^\mu(\tau, \sigma_i + 0)\right] = 0, \quad i = 1, \ldots, n - 1.$$

Substituting Eq. (14) into this relation we obtain the equations for the columns (16)

$$(\mathcal{A}_{i+1} - \mathcal{A}_i - h_i\mathcal{B}_i)\, \breve{S}_i = (\mathcal{B}_{i+1} - \mathcal{B}_i + h_i\mathcal{A}_i)\, \breve{C}_i. \tag{17}$$

Here and below we denote the constants

$$h_i = \frac{\omega}{Q_i} = \frac{\omega m_i}{\gamma} \left[\dot{X}^2(\tau, \sigma_i) \right]^{-1/2}, \tag{18}$$

$$\breve{C}_i = \cos \omega \sigma_i, \quad \breve{S}_i = \sin \omega \sigma_i, \quad C_i = \cos \omega (\sigma_i - \sigma_{i-1}), \quad S_i = \sin \omega (\sigma_i - \sigma_{i-1}).$$

One can express the columns (16) $\mathcal{A}_{i+1}$, $\mathcal{B}_{i+1}$ of functions u and $\tilde{u}$ in the segment $[\sigma_i, \sigma_{i+1}]$ (between masses m_i and m_{i+1}) via the similar coefficients in the $[\sigma_{i-1}, \sigma_i]$:

$$\begin{aligned} \mathcal{A}_{i+1} &= (1 + h_i \breve{C}_i \breve{S}_i) \, \mathcal{A}_i + h_i \breve{S}_i^2 \mathcal{B}_i, \\ \mathcal{B}_{i+1} &= -h_i \breve{C}_i^2 \mathcal{A}_i + (1 - h_i \breve{C}_i \breve{S}_i) \, \mathcal{B}_i. \end{aligned} \tag{19}$$

Hence, al mentioned coefficients may be expressed via $\mathcal{A}_1$, $\mathcal{B}_1$.

Substituting expression (14) into the closure condition (5) and into the n-th boundary condition (10) and keeping in mind Eqs. (12), (13), we obtain the following relations for amplitudes:

$$b_0 = -\theta a_0, \tag{20}$$

$$\begin{aligned} \mathcal{A}_1 &= M_\theta (\breve{C} \mathcal{A}_n + \breve{S} \mathcal{B}_n), \\ \mathcal{B}_1 &= M_\theta [(\breve{C} - h_n \breve{S}) \, \mathcal{B}_n - (\breve{S} + h_n \breve{C}) \, \mathcal{A}_n]. \end{aligned} \tag{21}$$

Here $\breve{C} \equiv \breve{C}_n = \cos 2\pi \omega$, $\breve{S} \equiv \breve{S}_n = \sin 2\pi \omega$,

$$M_\theta = \begin{pmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta, \end{pmatrix}, \quad C_\theta = \cos 2\pi \theta \omega, \quad S_\theta = \sin 2\pi \theta \omega.$$

The system of matrix equations (19), (21) is homogeneous one. It can be reduced after excluding factors $\mathcal{A}_i$, $\mathcal{B}_i$ with $i = 2, 3, \ldots, n$ to the form

$$M_1 \mathcal{A}_1 = M_2 \mathcal{B}_1, \qquad M_3 \mathcal{A}_1 = M_4 \mathcal{B}_1, \tag{22}$$

where matrices M_k are linear combinations of M_θ and the identity matrix I. In particular, for $n = 2$ they are

$$\begin{aligned} M_1 &= (\breve{C} - h_1 C_1 S_2) \, M_\theta - I, & M_2 &= -(\breve{S} - h_1 S_1 S_2) \, M_\theta, \\ M_3 &= (\breve{S} + h_1 C_1 C_2) \, M_\theta + h_2 I, & M_4 &= (\breve{C} - h_1 S_1 C_2) \, M_\theta - I. \end{aligned}$$

Taking into account mutual commutativity of the matrices M_k and excluding the column $\mathcal{B}_1$ (or $\mathcal{A}_1$) from the system (22) we obtain the system equivalent to Eqs. (22)

$$M \mathcal{A}_1 = 0, \qquad M \mathcal{B}_1 = 0. \tag{23}$$

Here the matrix $M = M_1 M_4 - M_2 M_3$ may be reduced with using equality $M_\theta^2 = 2 C_\theta M_\theta - I$. The system (23) (or (22)) has nontrivial solutions if and only if $\det M = 0$ or

$$2(C_\theta - \breve{C}) + \breve{S} \sum_{i=1}^n h_i - \sum_{i<j} h_i h_j s_{ji} s_{ij} + \sum_{i<j<k} h_i h_j h_k s_{ji} s_{kj} s_{ik} - \ldots + (-1)^{n+1} \prod_{i=1}^n h_i S_i = 0, \tag{24}$$

where $s_{ji} = \begin{cases} \sin \omega (\sigma_j - \sigma_i), & j > i, \\ \sin \omega (2\pi + \sigma_j - \sigma_i), & j < i, \end{cases}$. Eq. (24) connects unknown (for the present moment) values of parameters ω, θ, σ_i, Q_i, h_i.

In the case $n = 2$ equation (24) is

$$2(C_\theta - \breve{C}) + (h_1 + h_2)\,\breve{S} - h_1 h_2 S_1 S_2 = 0, \tag{25}$$

This equation may be rewritten after expanding notations

$$2(\cos 2\pi\theta\omega - \cos 2\pi\omega) + (h_1 + h_2)\,\omega \sin 2\pi\omega = = h_1 h_2 \sin\omega\sigma_1 \cdot \sin\omega(2\pi - \sigma_1).$$

Other relations connecting these parameters we obtain after substituting expression (14) into the orthonormality conditions (6):

$$\omega^2(A_i^2 + B_i^2 + \tilde{A}_i^2 + \tilde{B}_i^2) = a_0^2(1 + \theta^2), \qquad i = 1, \ldots, n; \tag{26}$$
$$\omega^2(\tilde{A}_i B_i - A_i \tilde{B}_i) = a_0^2\theta, \qquad i = 1, \ldots, n. \tag{27}$$

Among n equations (27) only one is independent, for example, with $i = 1$. If it is satisfied and the relations (19) take place, other conditions (27) is satisfied too. But n equations (26) are independent. Below we use the first of them and their residuals

$$\breve{C}_i(h_i\breve{C}_i + 2\breve{S}_i)(A_i^2 + \tilde{A}_i^2) + \breve{S}_i(h_i\breve{S}_i - 2\breve{C}_i)(B_i^2 + \tilde{B}_i^2) = 2(\breve{C}_i^2 - \breve{S}_i^2 - h_i\breve{C}_i\breve{S}_i)(A_i B_i + \tilde{A}_i\tilde{B}_i). \tag{28}$$

Here Eqs. (19) are used.

Under condition (25) the matrix $M = 0$ in Eq. (23) and an arbitrary nonzero column $\mathcal{A}_1$ or $\mathcal{B}_1$ is its eigenvector. It is connected with the rotational symmetry of the problem. So one can express, for example, the column $\mathcal{B}_1$ via $\mathcal{A}_1$ (the latter may be taken arbitrarily):

$$B_1 = \frac{-C_* A_1 + S_\theta \tilde{A}_1}{S_*}, \quad \tilde{B}_1 = -\frac{S_\theta A_1 + C_* \tilde{A}_1}{S_*}. \tag{29}$$

Here

$$C_* = \breve{C} - C_\theta - h_1 C_1(\breve{S}C_1 - S_1\breve{C} - h_2 S_2 S_3) - h_2\breve{C}_2 S_3,$$
$$S_* = \breve{S} - h_1 S_1(\breve{S}C_1 - S_1\breve{C} - h_2 S_2 S_3) - h_2\breve{S}_2 S_3, \qquad \text{for } n = 3,$$

$$C_* = \breve{C} - C_\theta - h_1 C_1 S_2, \quad S_* = \breve{S} - h_1 S_1 S_2, \qquad \text{for } n = 2.$$

Coefficients (29) must satisfy equations (26)–(28), resulting from the orthonormality conditions (6). After substitution expressions (29) into Eqs. (26), (27) with $i = 1$ we have

$$\omega^2 S_*^{-2}(C_*^2 + S_*^2 + S_\theta^2)(A_1^2 + \tilde{A}_1^2) = a_0^2(1 + \theta^2), \tag{30}$$
$$\omega^2 S_*^{-1} S_\theta(A_1^2 + \tilde{A}_1^2) = a_0^2\theta. \tag{31}$$

If we exclude factors $A_1^2 + \tilde{A}_1^2$ and a_0^2 from this system, we obtain

$$\frac{1 + \theta^2}{\theta} = \frac{C_*^2 + S_*^2 + S_\theta^2}{S_* S_\theta}. \tag{32}$$

This equation determines values of the parameters ω and θ. In the case $n = 2$ equation (32) takes the form

$$\frac{1 + \theta^2}{\theta} = \frac{2\breve{S} + (h_1 + h_2)\breve{C} - h_1 h_2 C_1 S_2}{S_\theta}. \tag{33}$$

To determine values $\sigma_1, \ldots, \sigma_{n-1}$ one should add $n - 1$ equations (28) to the system (24), (32) and take into account Eqs. (29).

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In the case $n = 2$ one equation (28) is reduced to the simple form

$$\sin\left[2\omega(\pi - \sigma_1)\right] = 0.$$

It determines a set of acceptable values σ_1:

$$\sigma_1 = \pi + \frac{\pi k}{2\omega}, \quad k \in Z, \quad |k| < 2\omega. \tag{34}$$

If values ω, θ, σ_i satisfy equations (24), (28), (32), the expression (14) satisfies the system (5)–(10) and describes an uniform rotation of the closed string with massive points (rotational state). The shape of this string is a section $t = t_0 = \text{const}$ of the world surface (14). This shape is the closed curve, composed from segments of a hypocycloid if and only if the equalities (24), (32) are fulfilled. This result is similar to the behavior of rotational states for the string baryon model "triangle" [10].

Hypocycloid is the curve drawing by a point of a circle (with radius r) rolling inside another fixed circle with larger radius R. In the case of solutions (14) uniformly rotating hypocycloidal segments of the string are joined at non-zero angles in the massive points. The relation of the mentioned radii is

$$\frac{r}{R} = \frac{1 - |\theta|}{2}.$$

For solutions (14) $|\theta| < 1$.

This hypocycloidal string rotates in the e_1, e_2 plane at the angular velocity $\Omega = \omega/a_0$, the massive points move at the speeds v_i along the circles with radii v_i/Ω. These values are connected by the following equations, resulting from Eqs. (13):

$$a_0 = \frac{m_1 Q_1}{\gamma \sqrt{1 - v_1^2}} = \ldots = \frac{m_n Q_n}{\gamma \sqrt{1 - v_n^2}}. \tag{35}$$

Speeds v_i are determined by Eqs. (13) and in the case $n = 2$ are equal

$$v_1^2 = \theta \frac{\breve{S} - h_2 S_1 S_2}{S_\theta}, \qquad v_2^2 = \theta \frac{\breve{S} - h_1 S_1 S_2}{S_\theta}. \tag{36}$$

The rotating string may also have cusps (return points) of the hypocycloid moving at the speed of light.

Values ω and θ are determined from the system (24), (32). Solution of the system (24), (32) (pairs ω, θ) form some countable set. Each pair corresponds to solution (14) describing uniform rotation of the closed string with certain topological type.

The rotational states (14) in the case $\theta \neq 0$ we shall name "hypocycloidal states".

In the case when the parameter in Eq. (12) equals zero ($\theta = 0$), solutions (14) describe rotational motions of n times folded string. It has a form of rotating rectilinear segment. These motions are divided into two classes: (a) "linear states" with all masses m_i moving at nonzero velocities v_i at the ends of the rotating rectilinear folded string, and (b) "central states" with one massive point (or some of them) placed at the rotational center (Fig. 2).

There are many topologically different types of linear, central and hypocycloidal states (14). They may be classified with the number of cusps and the type of intersections of the hypocycloid following Ref. [10]. Note that in the considered model (4) the string does not interact with itself in a point of intersection.

These topological configurations of the rotational states may be classified by investigation of the massless $m_i \to 0$ or ultrarelativistic $v_i \to 1$ limit for fixed γ and a_0. Analysis of equations (24), (32)–(36) shows that in the limit $m_i \to 0$ the values Q_i tend to infinity, values 2ω and $2\theta\omega$ tend to following integer numbers:

$$n_1 = \left| \lim_{m_i \to 0} 2\omega \right|, \qquad n_2 = \lim_{m_i \to 0} 2\theta\omega. \tag{37}$$

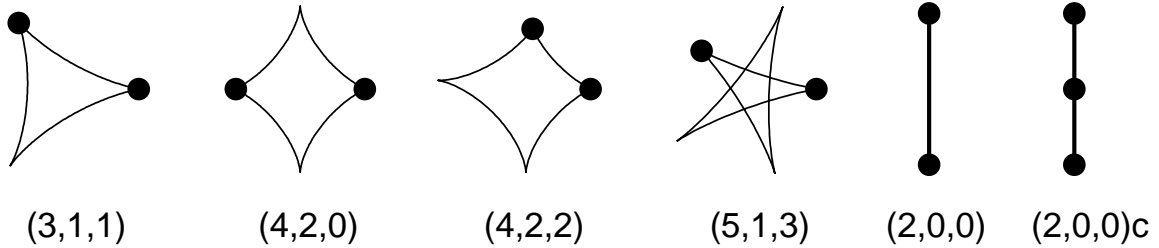| (3,1,1) | (4,2,0) | (4,2,2) | (5,1,3) | (2,0,0) | (2,0,0)c |

Figure 2: Examples of rotational states of the type (n_1, n_2, k)

In the first 5 examples the string carries $n = 2$ massive points, in the last (the right) case the central state with $n = 3$ is shown.

The left state in Fig. 2 in the limit $m_i \to 0$ tends to hypocycloid with $n_1 = 3$ cusps, for the 2-nd and 3-rd states this number is $n_1 = 4$ (they tend to astroid), but these curves have different numbers k in accordance with different positions of masses. The fourth state with $n_1 = 5$, $n_2 = 1$ correspond to the curvilinear star. The mentioned 4 examples present hypocycloid states.

The fifth example with the type $(2, 0, 0)$ describes the simplest linear rotational state with $n = 2$. In the limit $m_i \to 0$ this state and the central state (the rightmost in Fig. 2) tend to the same limit: the double rectilinear segment.

For linear and central rotational states we can put $\tilde{A}_1 = 0$ (without loss of generality) in the column $\mathcal{A}_1$. Equations $S_\theta = 0$, (19), (20), (29) result in equalities $\tilde{B}_1 = 0$, $b_0 = 0$

and $\tilde{u}(\sigma) = 0$. So the parametrization (14) of the string world surface for linear and central rotational states may be rewritten in the simple form

$$X^\mu(\tau, \sigma) = x_0^\mu + e_0^\mu a_0 \tau + u(\sigma) \cdot e^\mu(\omega\tau). \tag{40}$$

Parameters in Eq. (40) are determined from Eqs. (19)–(35), but we are to note that for the case $\theta = 0$ equations (27) become the identities and equations (32) or (33) loose their sense (and should be replaced by the equation $\theta = 0$). The values ω, σ_i are determined from Eqs. (24), (28).

In the case $n = 2$ for linear states (40) the value σ_1 is determined from Eq. (34) as before (with arbitrary even number k), and equation (24) has the form (25). If we substitute in this equation $C_\theta = 1$ and equalities $S_2 = (-1)^k S_1 = S_1$, $2S_1^2 = 1 - \check{C}$, resulting from Eq. (34), we obtain for linear states

$$(h_1 h_2 - 4) \tan \pi\omega = 2(h_1 + h_2), \tag{41}$$

Speeds v_i of massive points for linear states are

$$v_i = 2(4 + h_i^2)^{-1/2}. \tag{42}$$

These relations result from Eqs. (26), (29), (34), (41) in the case $\theta = 0$.

The central rotational states with $\theta = 0$, $n_2 = 0$ are described by Eq. (40), but some massive points are placed at the rotational center. For example, consider the central state with $n = 3$ massive points (the rightmost in Fig. 2), where the mass m_3 is the center ($v_3 = 0$) and masses m_1, m_2 move at nonzero speeds v_1 and v_2. The equality $v_3 = 0$ results in the condition $u(0) = 0$. It is equivalent to the equality

$$A_1 = 0, \tag{43}$$

that forbids to use Eqs. (29) for this state.

So we express coefficients A_2, B_2, A_3, B_3 via B_1 from Eqs. (19), for example, $A_3 = (h_1 S_1^2 + h_2 \check{S}_2^2 - h_1 h_2 \check{S}_2 S_1 S_2) B_1$ and substitute these expressions into Eqs. (21). Keeping in mind Eq. (43), we obtain two equations, connecting values ω, σ_1, σ_2, h_1, h_2. These equations after transformations take the form

$$S_1 + S_2 C_3 + C_2 S_3 = h_2 S_2 S_3, \qquad S_3 + S_1 C_2 + C_1 S_2 = h_1 S_1 S_2. \tag{44}$$

Equation (24) is the consequence of Eqs. (44).

Other relations between the mentioned values result from Eqs. (28) with $i = 1$ and $i = 2$. They may be reduced to the form

$$h_1 = 2\frac{C_1}{S_1} = 2 \cot \omega\sigma_1, \qquad h_2 = 2\frac{C_3}{S_3} = 2 \cot \omega(\pi - \sigma_1). \tag{45}$$

Eqs. (44) and (45) result in the equality

$$\sigma_2 - \sigma_1 = \pi, \tag{46}$$

and its consequences $C_2 = C_1 C_3 - S_1 S_3 = \cos \pi\omega$, $S_2 = S_1 C_3 + C_1 S_3 = \sin \pi\omega$, $\check{S} = 2S_2 C_2$.

Keeping in mind these relations we determine all coefficients of the function $u(\sigma)$:

$$A_2 = 2S_1 C_1 B_1, \quad B_2 = (S_1^2 - C_1^2)B_1, \quad A_3 = -\check{S}B_1, \quad B_3 = \check{C}B_1,$$

the value $a_0 = \omega B_1$ from Eq. (26), and, considering $\dot{X}^\mu$ at $\sigma = \sigma_1$, $\sigma = \sigma_2$ and Eq. (46), determine velocities of the massive points:

$$v_1 = S_1, \qquad v_2 = S_3. \tag{47}$$

These equalities and Eqs. (45) let us to express

$$h_i = 2\sqrt{v_i^{-2} - 1}, \qquad i = 1, 2$$

(coinciding with Eq. (42)) and, taking into account Eqs. (35), to obtain the equation

$$\frac{m_1 v_1}{1 - v_1^2} = \frac{m_2 v_2}{1 - v_2^2}. \tag{48}$$

If the initial data for this central rotational state are the values m_1, m_2, γ, v_1, than one can find v_2 from Eq. (48), and from Eqs. (45) the values h_1, h_2 and

$$\omega = \frac{1}{\pi}\left(\arctan\frac{2}{h_1} + \arctan\frac{2}{h_2}\right) + n_1^*; \qquad \sigma_1 = \frac{1}{\omega}\left(\arctan\frac{2}{h_1} + k\right), \tag{49}$$

and all other parameters of the world surface (40).

4. Stability problem for central rotational states

Possible applications of solutions (14) and (40) in hadron spectroscopy essentially depend on stability or instability of these states with respect to small disturbances. In this section we study spectrum of these disturbances for the central rotational states.

This problem has been recently solved for the closed string with $n = 1$ massive point for central states in Ref. [38], and for linear and hypocycloidal states in Ref. [39]. Here we generalize this approach to the case of larger numbers n ($n \leq 3$).

To solve the stability problem for rotational states (14) or (40) we consider the general solution of Eq. (8) for the string with n masses

$$X^\mu(\tau, \sigma) = \frac{1}{2}[\Psi_{i+}^\mu(\tau + \sigma) + \Psi_{i-}^\mu(\tau - \sigma)],] \qquad \sigma \in [\sigma_{i-1}, \sigma_i], \quad i = 1, \dots, n. \tag{50}$$

Here the functions $\Psi_{i\pm}^\mu(\tau \pm \sigma)$ are smooth, the world surface (50) is smooth between world lines of massive points.

We denote $\breve{\Psi}_{i\pm}^\mu$ the functions in the expression (50) for the rotational states (14) or (40). In particular, for the central rotational state (40) with $n = 3$ massive points (the rightmost in Fig. 2), where the mass m_3 is at the center, and equalities (43)–(49) take place, the derivatives of functions $\breve{\Psi}_{i\pm}^\mu$ are

$$\breve{\Psi}_{1\pm}^{\prime\mu}(\tau) = a_0\left[e_0^\mu \pm e^\mu(\omega\tau)\right],$$
$$\breve{\Psi}_{2\pm}^{\prime\mu}(\tau) = a_0\left[e_0^\mu + 2v_1 C_1 \acute{e}^\mu(\omega\tau) \pm (2v_1^2 - 1)\, e^\mu(\omega\tau)\right], \tag{51}$$
$$\breve{\Psi}_{3\pm}^{\prime\mu}(\tau) = a_0\left[e_0^\mu - \breve{S}\acute{e}^\mu(\omega\tau) \pm \breve{C}e^\mu(\omega\tau)\right],$$

To describe any small disturbances of the rotational motion, that is motions close to states (14) or (40) we consider vector functions $\Psi_{i\pm}^{\prime\mu}$ close to $\breve{\Psi}_{i\pm}^{\prime\mu}$ in the form

$$\Psi_{i\pm}^{\prime\mu}(\tau) = \breve{\Psi}_{i\pm}^{\prime\mu}(\tau) + \varphi_{i\pm}^\mu(\tau). \tag{52}$$

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The disturbance $\varphi_{i\pm}^{\mu}(\tau)$ is supposed to be small, so we omit squares of $\varphi_{i\pm}$ when we substitute the expression (52) into dynamical equations (5), (9) and (10). In other words, we work in the first linear vicinity of the states (14) or (40). Both functions $\Psi_{i\pm}^{\prime\mu}$ and $\breve{\Psi}_{i\pm}^{\prime\mu}$ in expression (52) must satisfy the condition

$$\Psi_{i+}^{\prime\,2} = \Psi_{i-}^{\prime\,2} = 0,$$

resulting from Eq. (6), hence in the first order approximation on $\varphi_{i\pm}$ the following scalar product equals zero:

$$(\breve{\Psi}_{i\pm}^{\prime}, \varphi_{i\pm}) = 0. \tag{53}$$

For the disturbed motions the equalities (11) $\sigma_i = $ const and (12) $\tau^* = \tau + 2\pi\theta$, generally speaking, is not carried out and should be replaced with the equalities

$$\sigma_1(\tau) = s_1 + \delta_1(\tau), \qquad \sigma_2(\tau) = s_2 + \delta_2(\tau), \qquad \tau^* = \tau + 2\pi\theta + \delta(\tau), \tag{54}$$

where $\delta_i(\tau)$ and $\delta(\tau)$ are small disturbances. In the case of the central states (51) $\theta = 0$.

Expression (52) together with Eq. (50) is the solution of the string motion equation (8). Therefore we can obtain equations of evolution for small disturbances $\varphi_{i\pm}^{\mu}(\tau)$, substituting expressions (52) and (54) with Eq. (51) into other equations of motion (9), (10), the closure condition (5) and the continuity condition

$$X^{\mu}(\tau, \sigma_i(\tau) - 0) = X^{\mu}(\tau, \sigma_i(\tau) + 0), \qquad i = 1, \ldots, n - 1. \tag{55}$$

We are to take into account nonlinear factors $\left\{\left[\frac{d}{d\tau} X(\tau, \sigma_i(\tau))\right]^2\right\}^{-1/2}$ and contributions from the disturbed arguments τ^* and $\sigma_i(\tau)$ (54), for example:

$$\breve{\Psi}_{n\pm}^{\prime\mu}(\tau^* \pm 2\pi) \simeq \breve{\Psi}_{n\pm}^{\prime\mu}(\tau + 2\pi\theta \pm 2\pi) + \delta(\tau)\,\breve{\Psi}_{n\pm}^{\prime\prime\mu}(\tau + 2\pi\theta \pm 2\pi).$$

This substitution for the central rotational state (40) with $n = 3$ and vector-functions $\breve{\Psi}_{i\pm}^{\mu}$ (51) after simplifying results in the following system of 6 vector equations in linear (with respect to $\varphi_{\pm}^{i\mu}$, δ_i and δ) approximation:

$$\varphi_{1+}^{\mu}(+_1) + \varphi_{1-}^{\mu}(-_1) - \varphi_{2+}^{\mu}(+_1) - \varphi_{2-}^{\mu}(-_1) + 4C_1 a_0[e^{\mu}(\omega\tau)\,\dot{\delta}_1(\tau) + \omega\acute{e}^{\mu}(\omega\tau)\,\delta_1] = 0,$$

$$\varphi_{2+}^{\mu}(+_2) + \varphi_{2-}^{\mu}(-_2) - \varphi_{3+}^{\mu}(+_2) - \varphi_{3-}^{\mu}(-_2) - 4C_3 a_0[e^{\mu}(\omega\tau)\,\dot{\delta}_2(\tau) + \omega\acute{e}^{\mu}(\omega\tau)\,\delta_2] = 0,$$

$$\varphi_{3+}^{\mu}(+) + \varphi_{3-}^{\mu}(-) - \varphi_{1+}^{\mu}(\tau) - \varphi_{1-}^{\mu}(\tau) + 2a_0 e_0^{\mu}\dot{\delta}(\tau) = 0,$$

$$\frac{d}{d\tau}\Big\{\varphi_{1+}^{\mu}(+_1) + \varphi_{1-}^{\mu}(-_1) + 2C_1 a_0(e^{\mu}\dot{\delta}_1 + \omega\acute{e}^{\mu}\delta_1) + F_1(e_0^{\mu} + v_1\acute{e}^{\mu})\Big\} +$$

$$+ Q_1\Big[\varphi_{1+}^{\mu}(+_1) - \varphi_{1-}^{\mu}(-_1) - \varphi_{2+}^{\mu}(+_1) + \varphi_{2-}^{\mu}(-_1)\Big] = 0. \tag{56}$$

$$\frac{d}{d\tau}\Big\{\varphi_{2+}^{\mu}(+_2) + \varphi_{2-}^{\mu}(-_2) - 2C_3 a_0(e^{\mu}\dot{\delta}_2 + \omega\acute{e}^{\mu}\delta_2) + F_2(e_0^{\mu} - v_2\acute{e}^{\mu})\Big\} +$$

$$+ Q_2\Big[\varphi_{2+}^{\mu}(+_2) - \varphi_{2-}^{\mu}(-_2) - \varphi_{3+}^{\mu}(+_2) + \varphi_{3-}^{\mu}(-_2)\Big] = 0.$$

$$\frac{d}{d\tau}\Big\{\varphi_{1+}^{\mu} + \varphi_{1-}^{\mu} + (\varphi_{1+} - \varphi_{1-})\,e_0^{\mu}\Big\} + Q_3\Big[\varphi_{3+}^{\mu}(+) - \varphi_{3-}^{\mu}(-) - \varphi_{1+}^{\mu} + \varphi_{1-}^{\mu} + 2\omega a_0\acute{e}^{\mu}\delta\Big] = 0.$$

Here arguments (τ) for $\varphi_{1\pm}^{\mu}$, δ, δ_i and $(\omega\tau)$ for e^{μ}, $\acute{e}^{\mu}$ may be omitted; we use the following notations for arguments

$$(\pm_1) \equiv (\tau \pm \sigma_1), \qquad (\pm_2) \equiv (\tau \pm \sigma_2), \qquad (\pm) \equiv (\tau \pm 2\pi),$$

for the scalar products

$$\varphi_{i\pm}^0 \equiv (e_0, \varphi_{i\pm}), \qquad \varphi_{i\pm}^3 \equiv (e_3, \varphi_{i\pm}), \qquad \varphi_{i\pm} \equiv (e, \varphi_{i\pm}), \qquad \acute{\varphi}_{i\pm} \equiv (\acute{e}, \varphi_{i\pm}) \qquad (57)$$

and

$$F_1 = \varphi_{1+}(+_1) - \varphi_{1-}(-_1) - v_1 C_1^{-1} \big[\acute{\varphi}_{1+}(+_1) + \acute{\varphi}_{1-}(-_1) - 2\omega a_0 \delta_1 \big],$$
$$F_2 = C_3^{-1} \big\{ \check{C}_2 [\varphi_{2-}(-_2) - \varphi_{2+}(+_2)] + \check{S}_2 [\acute{\varphi}_{2+}(+_2) + \acute{\varphi}_{2-}(-_2)] + 2\omega v_2 a_0 \delta_2 \big\}.$$

The first two equations (56) results from Eqs. (55), the third — from Eq. (5), other ones are consequence of Eqs. (9) and (10). Equations (56) are simplified with using Eqs. (43)–(49), (51) and equalities (53), resulting in the following relations for projections (57) of disturbances:

$$\varphi_{1\pm}^0(\tau) \pm \varphi_{1\pm}(\tau) = 0, \quad \varphi_{2\pm}^0 + 2v_1 C_1 \acute{\varphi}_{2\pm} \pm (2v_1^2 - 1)\,\varphi_{2\pm} = 0, \quad \varphi_{3\pm}^0 - \check{S}\acute{\varphi}_{3\pm} \pm \check{C}\varphi_{3\pm} = 0. \quad (58)$$

The linearized system of equations (56), (58) describes evolution of small disturbances of the considered central rotational state (40), (51).

Note that scalar products of Eqs. (56) onto the vector e_3 (orthogonal to the rotational plane e_1, e_2) form the closed subsystem from 6 equations with respect to 6 functions (57) $\varphi_{i\pm}^3$:

$$\varphi_{i+}^3(+_i) + \varphi_{i-}^3(-_i) = \varphi_{i*+}^3(+_i) + \varphi_{i*-}^3(-_i),$$
$$\varphi_{3+}^3(+) + \varphi_{3-}^3(-) = \varphi_{1+}^3(\tau) + \varphi_{1-}^3(\tau),$$
$$\dot{\varphi}_{i+}^3(+_i) + \dot{\varphi}_{i-}^3(-_i) + Q_i \big[\varphi_{i+}^3(+_i) - \varphi_{i-}^3(-_i) - \varphi_{i*+}^3(+_i) + \varphi_{i*-}^3(-_i)\big] = 0,$$
$$\dot{\varphi}_{1+}^3(\tau) + \dot{\varphi}_{1-}^3(\tau) + Q_3 \big[\varphi_{3+}^3(+) - \varphi_{3-}^3(-) - \varphi_{1+}^3(\tau) + \varphi_{1-}^3(\tau)\big] = 0. \qquad (59)$$

Here $i = 1, 2$, $i^* \equiv i + 1$. This system is homogeneous system with deviating arguments.

We search solutions of this system in the form of harmonics

$$\varphi_{j\pm 3} = B_{j\pm}^3 e^{-i\tilde{\omega}\tau}. \qquad (60)$$

This substitution results in the linear homogeneous system of 6 algebraic equations with respect to 6 amplitudes $B_{i\pm}^3$. The system has nontrivial solutions if and only if its determinant

$$\begin{vmatrix} E_{1+} & E_{1-} & -E_{1+} & -E_{1-} & 0 & 0 \\ 0 & 0 & E_{2+} & E_{2-} & -E_{2+} & -E_{2-} \\ -1 & -1 & 0 & 0 & E_{3+} & E_{3-} \\ (i\tilde{\omega} - Q_1)E_{1+} & (i\tilde{\omega} + Q_1)E_{1-} & Q_1 E_{1+} & -Q_1 E_{1-} & 0 & 0 \\ 0 & 0 & (i\tilde{\omega} - Q_2)E_{2+} & (i\tilde{\omega} + Q_2)E_{2-} & Q_2 E_{2+} & -Q_2 E_{2-} \\ -i\tilde{\omega} - Q_3 & -i\tilde{\omega} + Q_3 & 0 & 0 & Q_3 E_{3+} & -Q_3 E_{3-} \end{vmatrix} = 0$$

equals zero. Here $E_{j\pm} = \exp(\mp i\tilde{\omega}\sigma_j)$. This equation is reduced to the form

$$2(1 - \cos 2\pi\tilde{\omega}) + \tilde{\omega}(Q_1^{-1} + Q_2^{-1} + Q_3^{-1})\sin 2\pi\tilde{\omega} =$$

$$= \tilde{\omega}^2 \left(\frac{\sin^2 \pi\tilde{\omega}}{Q_1 Q_2} + \frac{\sin\tilde{\sigma}_3\tilde{\omega} \cdot \sin\sigma_2\tilde{\omega}}{Q_2 Q_3} + \frac{\sin\sigma_1\tilde{\omega} \cdot \sin\tilde{\sigma}_{23}\tilde{\omega}}{Q_1 Q_3} \right) - \frac{\tilde{\omega}^3 \sin\sigma_1\tilde{\omega} \cdot \sin\pi\tilde{\omega} \cdot \sin\tilde{\sigma}_3\tilde{\omega}}{Q_1 Q_2 Q_3}, \quad (61)$$

where $\tilde{\sigma}_3 = 2\pi - \sigma_2 = \pi - \sigma_1$, $\tilde{\sigma}_{23} = 2\pi - \sigma_1$. This equation coincides with Eq. (24) with $n = 3$, if ω is substituted by $\tilde{\omega}$. The spectrum of transversal (with respect to the e_1, e_2 plane) small fluctuations of the string for the considered rotational state contains frequencies $\tilde{\omega}$ which are roots of Eq. (61). Analysis of the real and imaginary parts of this equation demonstrates that

all these frequencies are real numbers, therefore amplitudes of such fluctuations do not grow with growth of time t.

Let us consider small disturbances concerning to the e_1, e_2 plane. Projections (scalar products) of equations (56) onto 3 vectors e_0, $e(\tau)$, $\acute{e}(\tau)$ form the system of 18 differential equations with deviating arguments with respect to 15 unknown functions of the argument τ: $\varphi_{j\pm}$, $\acute{\varphi}_{j\pm}$ $(j = 1, 2, 3)$, δ_1, δ_2, δ (functions $\varphi_{j\pm}^0$ are excluded via Eqs. (58)).

When we search solutions of this system in the form of harmonics (60)

$$\varphi_{j\pm}^0 = B_{j\pm}^0 e^{-i\tilde{\omega}\tau}, \qquad \varphi_{j\pm} = B_{j\pm} e^{-i\tilde{\omega}\tau}, \qquad \acute{\varphi}_{j\pm} = \acute{B}_{j\pm} e^{-i\tilde{\omega}\tau}, \qquad 2a_0\delta_j = \Delta_j e^{-i\tilde{\omega}\tau}, \qquad (62)$$

we obtain the homogeneous system of 18 algebraic equations with respect to 15 amplitudes $B_{j\pm}^0$, $B_{j\pm}$, $\acute{B}_{j\pm}^0$, Δ_1, Δ_2, Δ. Three of these 18 equations are linear combinations of other ones.

For the rest 15 equations we use the mentioned above condition of existence of nontrivial solutions for this system. It is vanishing the corresponding determinant. These equations and calculations are cumbersome, so we omit this system and present here the result of symbolic calculation in the package MATLAB. In the case $m_1 = m_2$, $\sigma_1 = \tilde{\sigma}_3 = \pi/2$ the condition of vanishing this determinant is reduced to the following equation:

$$4Q_3^2 \tan^2 \pi\tilde{\omega} + 4Q_3 \left(\tilde{\omega} + \frac{\omega^2}{2\tilde{\omega}} \right) \tan \pi\tilde{\omega} + \tilde{\omega}^2 - \omega^2 = 0. \qquad (63)$$

This equation generalizes the condition in Ref. [38] for the closed string with $n = 1$ massive point. It transforms into the mentioned condition in the limit $m_1 \to 0$, $m_2 \to 0$. The transcendental equation (63) contains a denumerable set of real roots (frequencies). They correspond to different modes of small oscillations of the string in the considered central rotational state (40).

This state will be unstable, if there are complex frequencies $\tilde{\omega} = \check{\omega} + i\xi$ in the spectrum, generated by Eq. (63). If its imaginary part ξ will be positive, the modes of disturbances φ^μ (corresponding to the root $\check{\omega} + i\xi$) get the multiplier $\exp(\xi\tau)$, that is they grow exponentially.

The search of complex roots of equation (63) in Ref. [38] showed that such roots can exist only on the imaginary axis of the complex plane $\tilde{\omega}$. On this axis of $\tilde{\omega}$ (in the case $\tilde{\omega} = i\xi$) the equation (63) takes the form

$$4Q_3^2 \tanh^2 \pi\xi + \xi^2 + \omega^2 = 4Q_3 \left(\frac{\omega^2}{2\xi} - \xi \right) \tanh \pi\xi. \qquad (64)$$

The left hand side of this equation grows with growing ξ (for $\xi > 0$), and the right hand side decreases. It is obvious (see the limit $\xi \to 0$), that the root $\xi > 0$ of Eq. (64), that is the imaginary root $\tilde{\omega} = i\xi$ of Eq. (63) exists, if and only if

$$2\pi Q_3 > 1. \qquad (65)$$

If we use the expression (35) in the form $Q_3 = \gamma a_0/m_3$ (remind that $v_3 = 0$ for this central state) we reduce the criterion (65) to the following form:

$$m_3 < m_{cr} \equiv 2\pi\gamma a_0. \qquad (66)$$

Thus, we obtain the threshold effect in stability properties of the central rotational states under consideration. If the central mass m_3 is greater than m_{cr}, hence all roots of Eq. (63) are

real ones and the state is stable. But in the case $m_3 < m_{cr}$ the state is unstable: the imaginary root $\tilde{\omega} = i\xi$ appears and the corresponding amplitude of disturbances grows exponentially: $\varphi = Be^{\xi\tau}$.

This threshold effect or the spontaneous symmetry breaking for the string state was observed in numerical experiments in Ref. [38]. Note that our analysis of small disturbances is suitable only for initial stage of an unstable motion when disturbances are really small.

In the following section the stable rotational states are applied in hadron spectroscopy.

5. Regge trajectories

The obtained rotational motions of the considered model should be applied for describing physical manifestations of glueballs and other exotic particles, in particular, their Regge trajectories. For this purpose we calculate the energy E and classic angular momentum L for the states (14) of this model. For an arbitrary classic state of the relativistic string with the action (4) carrying massive points they are determined by the following integrals (Noether currents) [10, 37]:

$$P^\mu = \int_{\mathcal{C}} p^\mu(\tau,\sigma)\, d\sigma + \sum_{i=1}^n p_i^\mu(\tau), \tag{67}$$

$$\mathcal{L}^{\mu\nu} = \int_{\mathcal{C}} \left[X^\mu(\tau,\sigma)\, p^\nu(\tau,\sigma) - X^\nu(\tau,\sigma)\, p^\mu(\tau,\sigma) \right] d\sigma + \sum_{i=1}^n \left[x_i^\mu(\tau)\, p_i^\nu(\tau) - x_i^\nu(\tau)\, p_i^\mu(\tau) \right], \tag{68}$$

where $x_i^\mu(\tau) = X^\mu(\tau, \sigma_i(\tau))$ and $p_i^\mu(\tau) = m_i \dot{x}_i^\mu(\tau)/\sqrt{\dot{x}_i^2(\tau)}$ are coordinates and momentum of the massive points, $p^\mu(\tau,\sigma) = \gamma[(\dot{X},X')X'^\mu - X'^2\dot{X}^\mu]/\sqrt{-g}$ is the canonical string momentum, $\mathcal{C}$ is any closed curve (contour) on the tube-like world surface of the string. Note that the lines $\tau = \mathrm{const}$ on the world surface (14) are not closed in the case $\tau_0 \neq 0$. So we can use the most suitable lines $\tau - \theta\sigma = \mathrm{const}$ (that is $t = \mathrm{const}$) as the contour C in integrals (67), (68).

The reparametrization $\tilde{\tau} = \tau - \theta\sigma$, $\tilde{\sigma} = \sigma - \theta\tau$ keeps the orthonormality conditions (6). Under them $p^\mu(\tau,\sigma) = \gamma\dot{X}^\mu(\tau,\sigma)$.

The square of energy E^2 equals the scalar square of the conserved vector of momentum (67): $P^2 = P_\mu P^\mu = E^2$. If we substitute the expressions (14), (19), (20), (29)–(31) into Eq. (67) we obtain the following formula for the momentum:

$$P^\mu = e_0^\mu E, \qquad E = 2\pi\gamma a_0(1 - \theta^2) + \sum_{i=1}^n \frac{m_i}{\sqrt{1 - v_i^2}}. \tag{69}$$

For the classical angular momentum (68) only z-component of $\mathcal{L}^{\mu\nu}$ is nonzero:

$$\mathcal{L}^{\mu\nu} = \ell_3^{\mu\nu} L, \qquad L = \frac{\gamma a_0^2}{2\omega}\left[2\pi(1 - \theta^2) + \sum_{i=1}^n \frac{v_i^2}{Q_i} \right]. \tag{70}$$

Here $j_3^{\mu\nu} = e_1^\mu e_2^\nu - e_1^\nu e_2^\mu = e^\mu \acute{e}^\nu - e^\nu \acute{e}^\mu$.

One can obtain the total angular momentum $J = L + S$ from the classical momentum (70) after quantization the system. But this problem for the string with masses (4) is not solved yet because of essential nonlinearity of equations (9), (10). So we use below the approach, suggested in Refs. [6, 5] for string models of mesons and baryons. It includes the spin contribution to the classical angular momentum in the form

$$J = L + S, \qquad S = \sum_{i=1}^n s_i, \tag{71}$$

where s_i are spin projections of massive points (valent glueballs), and also the following contribution to the energy (69) because spin-orbit interaction [5]:

$$\Delta E_{SL} = \sum_{i=1}^{n} [1 - (1 - v_i^2)^{1/2}](\Omega \cdot s_i). \tag{72}$$

Below we suppose that the value S in Eq. (71) corresponds to the maximal total momentum (71), that is $S = 2$ for 2-gluon glueballs [21]. Other values of model parameters are:

$$\gamma = 0.175 \text{ CeV}^2, \quad m_1 = m_2 = 700 \text{ MeV}. \tag{73}$$

This tension γ corresponds to the slope of Regge trajectories (1) for hadrons $\alpha' \simeq 0.9 \text{ GeV}^{-2}$. Estimations of gluon masses on the base of gluon propagator [40], in particular, in lattice calculations [41, 42] yield values m_i from 700 to 1000 MeV.

If the values m_i, γ and the topological type (n_1, n_2, k_j) of the rotational state (14) are fixed we obtain the one-parameter set of motions with different values E and J. These states lay at quasilinear Regge trajectories. As the parameter of this set one can use any of these values: ω, θ, a_0, E, J et al. Other values may be expressed from Eqs. (25) – (36).

In particular, in the case $n = 2$ and $m_1 = m_2$ (important for applications for glueballs) the equalities $h_1 = h_2$, $v_1 = v_2$ take place, and the system (25), (33) is reduced to one equation

$$S_1^2 \Sigma_\theta^2 + 2(S_2 + S_1 C_\theta)(C_1 \Sigma_\theta - S_2) = C_1^2 S_\theta^2, \tag{74}$$

where $\Sigma_\theta = \frac{1}{2} S_\theta (1 + \theta^2)/\theta$.

For every given value ω and fixed k (this lets us to obtain σ_1 from Eq. (34) and also S_i, C_i) we find θ from equation (74), then the values h_i, a_0, v_i from Eqs. (25), (33), (35), (36). Using Eqs. (69) – (72) we obtain the dependence $J = J(E^2)$ (the Regge trajectory).

Regge trajectories, calculated for rotational states (14), (40) of closed string with 2 massive points with different topological types (see Fig. 2) are shown in Fig. 3 with the corresponding type (n_1, n_2, k). For all curves the values of parameters $S = 2$ and (73) are chosen, J is in units $\hbar$. The pomeron trajectory (3) is shown as the dashed line.

These Regge trajectories are nonlinear for small E and tend to linear if $E \to \infty$. Their slope in this limit depends on the fixed topological type.

The ultrarelativistic limit $E \to \infty$ corresponds to $v_i \to 1 - 0$ (except for central states) and for values ω and θ — to the limits (37). Substituting into Eqs. (25), (32), (36), (69), (71) asymptotic relations with small values $\varepsilon_1 = \sqrt{1 - v_1^2}$, $\varepsilon_2 = \sqrt{1 - v_2^2}$, $2\omega = n_1 - \varepsilon_\omega$, $n_1 \theta = n_2 - \varepsilon_\theta$, we obtain in the limit $J \to \infty$, $E \to \infty$ the following asymptotic relation between these values for fixed type (n_1, n_2, b_k) of the state:

$$J \simeq \alpha' E^2 + \alpha_1 E^{1/2} + \alpha_2 E^{-1/2}, \qquad E \to \infty, \tag{75}$$

where

$$\alpha' = \frac{1}{2\pi\gamma} \frac{n_1}{n_1^2 - n_2^2}, \tag{76}$$

$$\alpha_1 = -\frac{\sqrt{2} \, n_1 (\sum m_i^{3/2})}{3\sqrt{\pi}\gamma(n_1^2 - n_2^2)^{3/4}}, \qquad \alpha_2 = \sqrt{\frac{\pi}{2}}(n_1^2 - n_2^2)^{1/4} \sum_{i=1}^{n} s_i m_i^{3/2}.$$

This dependence is close to linear one (1), but the slope α' (76) for this system differs from Nambu value $\alpha' = 1/(2\pi\gamma)$ by the factor
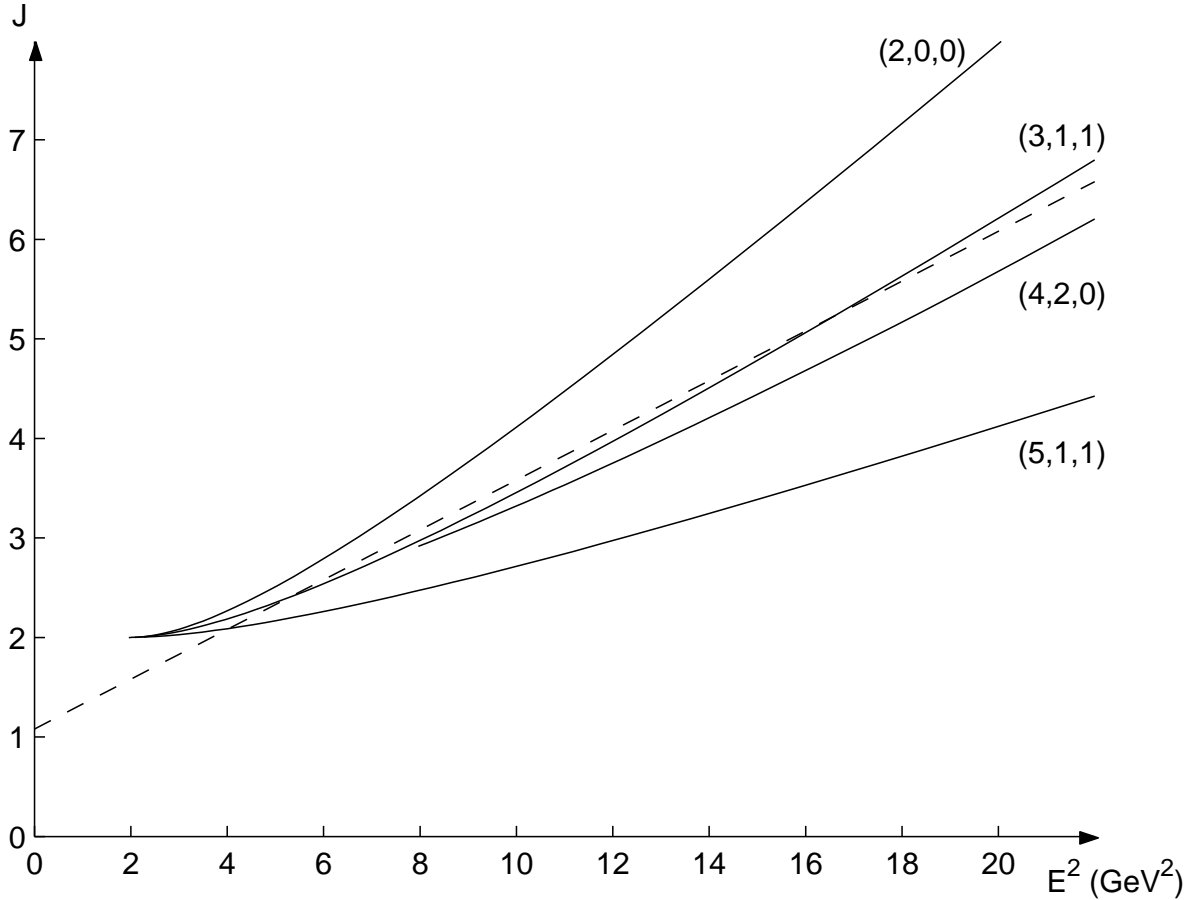
$$\chi = \frac{n_1}{n_1^2 - n_2^2}. \tag{77}$$

Figure 3: Regge trajectories for rotational states (14) with different topological types

In particular, the maximal slope with the factor $\chi = 1/2$ corresponds to the linear state with the type $(2, 0, 0)$ (two masses connected two strings without singularities). This state's trajectory has the slope (76) $\alpha' \simeq 0.45$ GeV^{-2}. It is larger that the slope $\alpha' \simeq 0.25$ of the pomeron trajectory (3) (the dashed line in Fig. 3). These trajectories distinctly diverge in Fig. 3.

The Regge trajectories for central states are not shown in Fig. 3 because of their instability, studied in Sect. 4. This instability takes place in the case (66) that corresponds to $E > 3m_i$.

For chosen values (73) of gluon masses the most close to the pomeron (glueball) trajectory (3) is the trajectory for the "triangle" configuration $(3, 1, 1)$. For this state $\chi = 3/8$ and the slope (76) $\alpha' \simeq 0.337$ GeV^{-2}. It is a bit larger than the value (3), so at very high energies E these trajectories diverge. Some other types of rotational states also generate suitable Regge trajectories, for example, the state with $n_1 = 4$, $n_2 = 2$ gives $\alpha' \simeq 0.3$.

Conclusion

The obtained rotational states (14) of the closed string with n massive points are divided in 3 groups: hypocycloidal, linear and central states, and also in a set of different topological classes, described by the integer parameters (37) $(n_1, n_2, k_1, \ldots, k_{n-1})$. The states from these classes generate the wide spectrum of quasilinear Regge trajectories (69), (71) with different slopes (76) in the limit of large energies. This slope α' (76) depends on only numbers (37) n_1, n_2, describing the limiting shape of the closed string in the limit $m_i \to 0$ and does not depend on mutual positions of massive points (numbers k_j).

For the central rotational states with the mass m_3 at the rotational center and moving masses $m_1 = m_2$ the stability with respect to small disturbances is investigated. It is shown that these states are unstable, if the central mass is less than the critical value (66). In this case the spectrum disturbances has exponentially growing modes.

Regge trajectories (69), (70) for rotational states (14) states were calculated with spin corrections in the form (71), (72). There are some classes of hypocycloidal rotational states (14) suitable for describing the pomeron (glueball) trajectory (3), in particular the state with $n_1 = 3$, $n_2 = 1$.

The considered model needs further development, in particular, quantization or quantum corrections. These corrections are to be significant for calculation of the intercept α_0.

Acknowledgment

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